

Average Path Length in Complex Networks: Patterns and Predictions

Reginald D. Smith

Bouchet-Franklin Research Institute, P.O. Box 10051 ,Rochester, NY 14610

E-mail: rsmith@sloan.mit.edu

Abstract. A simple and accurate relationship is demonstrated that links the average shortest path, nodes, and edges in a complex network. This relationship takes advantage of the concept of link density and shows a large improvement in fitting networks of all scales over the typical random graph model. The relationships herein can allow researchers to better predict the shortest path of networks of almost any size.

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The research of complex networks has exploded over the past decade with literally thousands of papers describing and theorizing about such networks in all details. This explosion of research followed the widespread availability of large network databases aided by the advance of computer technology and widespread online applications used by millions of users. Among the most prominent and well-known studies have been those of the Internet [1], metabolic pathways[2], and scientific collaborations[3, 4]. Other networks have also included sexual contacts[5], instant messaging[6], Congressional committees[7], jazz musicians[8], blogs[9], airports[10], and rappers [11].

Several review articles have highlighted the main features and characteristics of complex networks [12, 13, 14]. One of the most studied and important features of a complex network as been found to be the average path length (or characteristic path), \bar{l} that characterizes a network. It describes the average number of links that form the shortest path between any two nodes in the network. This property, more than any other, gives rise to what is known as "small world" behavior.

1. Brief Properties of \bar{l}

In their seminal work that helped ignite research into small world phenomena, Watts and Strogatz [15] describe small world networks as those which are connected, where the number of nodes is much larger than the average degree per node, and the average path length scales with $\log n$. Though random graphs can exhibit small world behavior, most graphs in the real world are not random and are often distinguished from random graphs by a relative high degree of clustering among nodes as measured by the clustering coefficients. Watts and Strogatz also described an estimate from random graph theory for the average path length of a random graph, which has become very useful for comparison with real networks,

$$\bar{l} \approx \frac{\ln N}{\ln \langle k \rangle} \quad (1)$$

where N is the number of nodes and k is the average degree per node in the network which is E/N for directed networks and $2E/N$ for undirected networks where E is the number of links (edges) in the graph. This equation gives a very good approximation for many networks and though it is not exact, it usually gives a good rough estimate. However, as an approximation it is usually only used to compare the average path length of a graph using real or simulated data and a similar random graph with the same N and $\langle k \rangle$.

There has been much more work done on \bar{l} describing its theoretical relationship with the small world network it characterizes [16, 17, 18, 19, 20]. Small-world networks have been analyzed using percolation theory and mean field theory among others to attempt to understand the exact nature of the transition from a "large" to a small world network. Since \bar{l} is one of the key parameters that signifies such a change, its theoretical relationship has been investigated in order to relate it to other properties of complex networks such as the correlation length of the network.

2. Link Density and Average Path Length

Though the random graph approximation is useful, it can be asked whether there is a better model for complex networks that can explain known data. Complex networks,

despite having similar average path lengths or clustering coefficients can vary in other measures such as first order degree distributions, assortative (or disassortative) mixing, and sizes of connected components. Given the many important topological features, much less the feedback with the dynamics on the network that affects network evolution, it can be questioned whether any more precise generalization is possible among complex networks.

One key concept that can link many disparate graphs, despite their number of nodes, is the concept of network density. Network density has been described in some papers[21, 22]. The definition used here is the ratio of the number of edges in the network over the total possible number of edges in the complete graph

$$\bar{a} = \frac{2E}{N(N-1)} \quad (2)$$

The network density has a maximum of 1 in a complete graph and a minimum of $2/(N-1) \approx 2/N$ in a simple ring topology. This density is also identical to the value of p , the probability two nodes will be connected by a link. However, it will be referred to as density throughout this paper since this paper does not concentrate on aspects of probability or percolation theory. This density will be used for both directed and undirected networks. For simplification, since the number of nodes in a network is usually $N \gg 1$ the link density can also be approximated as

$$\bar{a} = \frac{2E}{N^2} \quad (3)$$

However, the link density does not directly correlate with the average path length. Matching \bar{a} against \bar{l} shows very little relationship. Part of the problem is that with increasingly larger networks, a lower link density is sufficient to obtain a given average path length. In general, a larger network has a much smaller \bar{a} for a given \bar{l} than a smaller network with a similar \bar{l} .

Though the relationship between the two variables is tenuous, their product has several interesting properties

$$D = \bar{a}\bar{l} \quad (4)$$

D , is equal to 1 for both complete graphs ($E = N^2/2, \bar{l} = 1$) and directed ring topologies ($E = N, \bar{l} = N/2$) which as strongly connected clusters are networks with the longest possible average path. The undirected ring topologies have $\bar{l} = N/4$. However, outside these two extreme cases, D , typically does not equal 1 but has a much lower value, but greater than 0. The value of D varies much with \bar{a} so also has a large dependence on the size of the network with \bar{l} having a minimal impact.

A way to resolve the issue is to find a method of normalizing the network density so it is comparable across networks of all sizes. I define a normalized network density, \bar{a}_s , by taking the logarithm of the network density with a base of $N/2$ and adding 1 which is equivalent to

$$\bar{a}_s = 1 + \frac{\log \frac{2E}{N^2}}{\log N/2} \quad (5)$$

where \log here designates a natural logarithm. When the network is a complete network the normalized network density is 1, while for ring topology, it equals 0. Therefore, the size of the network will not affect the minimum network density.

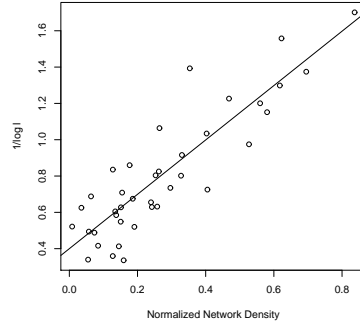


Figure 1. Plots of $\frac{1}{\log \bar{l}}$ vs. normalized network density. The slope of the fit is 1.5 respectively with an intercept of 0.4. R^2 of 0.78

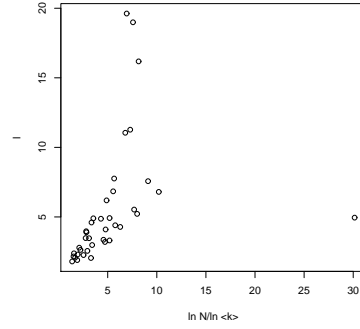


Figure 2. Plots of the same network data with \bar{l} vs. $\frac{\ln N}{\langle k \rangle}$. The data fits relatively well for graphs with small average path lengths or small N but shows greater disparities when these conditions are not met.

Not only does this normalized network density allow you to compare network densities over networks of various sizes, it actually demonstrates a correlation with the path length of the network, in particular the inverse of $\log \bar{l}$.

The graph in Figure 1 was developed using data from 39 different networks described in various papers. The values of these networks are shown in Appendix I. These networks are of many different types and have been given broad categorizations following those used by Newman[12].

This relationship was experimentally discovered and not quite expected. In fact, one of its more interesting properties is how it fits disparate networks, of all scales and average path lengths, accomodating the data better than the random graph estimation in Figure 2

In fact in Figure 1 the main points that do not fit well to the least squares line are those from biological networks, including the food webs (freshwater and marine) and metabolic networks. This may indicate either the data on these networks or incomplete or the underlying organizational property driving this relationship is less

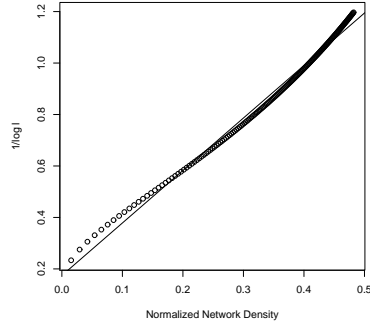


Figure 3. Plots of a graph of 1,000 nodes from 1,100 to 20,000 nodes where $\bar{l} = \frac{\ln N}{\ln \langle k \rangle}$

active in biological networks. The relation derived from the regressions implies

$$m\bar{a}_s + C = \frac{1}{\log \bar{l}} \quad (6)$$

this allows us to relate \bar{l} to the normalized density with the equation

$$\bar{l} = e^{\frac{1}{m\bar{a}_s + C}} \quad (7)$$

A quick but interesting example can be made using equation 7. Assuming the US population is 300M and accepting Milgram's six degrees of separation ($\bar{l} = 6$) we can estimate the average $\langle k \rangle$ for the US population at 14.6. This is much less than the 25.9 estimated from random graph theory (and assumes us to be substantially less gregarious).

A key question about the relationship in Figure 1 is how widely it applies to all types of networks. All of the networks sampled are described by authors as having "scale-free" or "long-tailed" characteristics. Obviously, graph theory does not constrain a network from being of this type so by looking at the relationship using data from an artificial random graph we can begin to push the boundaries of its applicability.

In Figure 3 it is clearly visible that the linear relation among real networks also holds for random graph data.

The slope of the plot from a random graph approximation is 2.04 which is slightly steeper than the slope of data from real networks. Therefore, this relationship is likely widely held among many small-world networks with a variety of topologies though there are likely exceptions.

3. Discussion

First, it should be acknowledged that though this relationship seems to fit a wider variety of real networks than random graph theory, it is not perfect. From the standards of theoretical prediction, the statistical fit still allows much leeway for the relationship between the quantities plotted against each other. However, this

relationship does fit data more consistently over all size scales in real networks than the usual random graph theory treatment.

Despite the interesting relationship this data reveals, it also raises the question of what the parameters of the linear plot actually mean. One clue can be gleaned from looking at the rate of change of the average shortest path vs. the normalized link density $\frac{\partial \bar{l}}{\partial \bar{a}_s}$.

Given equation 6 you can easily deduce from the fact that

$$\frac{\partial \bar{l}}{\partial \bar{a}_s} = -m\bar{l}[\log \bar{l}]^2 \quad (8)$$

So the slope, m , can be seen as the constant of proportionality between the rate of change with increasing network density in the average shortest path and the average shortest path. In fact, Equation 8 gives quite intuitive solutions since $m > 0$. The larger \bar{l} is, the more rapidly you can reduce the average shortest path of the network by increasing the network density. This intuitively fits with the observation by Watts and Strogatz [15] that in relatively sparse topologies, shortcuts can drastically reduce the average path length of the network leading to small world behavior. As the network becomes more dense, such short cuts give incrementally smaller reduction in the network average path. When you reach a complete network at $\bar{l} = 1$ there is a fixed point given you have maximum density and can no longer reduce the diameter of the network.

Additionally, m could be some measure of a quantity such as the "mass" of the network. If $\log \bar{l}$ can somehow be seen as a length then this gives additional meaning to the normalized link density. Here m would be a characteristic mass in all networks that is distributed over a one-dimensional interval determined by $\log \bar{l}$ and \bar{a}_s measures the resultant length density. However, the question of what m really is as far as a value is still unanswered. The fact that it seems consistent across such a wide variety of networks suggests it is some constant, perhaps of a transcendental number or ratio. This is all speculative though. Until a firm theoretical underpinning for the above results is made, the exact value of m is still subject to speculation.

In addition, although this relation seems to hold across a wide variety of networks, there are obviously situations where equations such as equation 7 break down. For example, when \bar{a}_s is 0, a complete graph, \bar{l} should have a value of 1, however, this does not necessary flow from the relations shown here. The only exception is equation 8 that shows a fixed point at $\bar{l} = 0$ as is expected. Therefore, in the regions of nearly complete graphs or sparse graphs, possibly where $p < p_c$ where p_c is the critical probability from the percolation theory, this relationship does not reliably apply. However, these regions are not the domain of almost all real networks.

4. Conclusion

This paper has shown that there is an intrinsic relationship between the average path length \bar{l} and the normalized link density, related to the number of nodes and edges, that is present in all networks. This relationship fits well in both real networks which often have a scale-free or non-random character, but can also describe random networks as well and likely most small world networks. Given the breakdown of the theory near the complete and ring topologies it may be surmised this only applies to graphs with small-world character that have a link probability $p_c < p < 1$. Much more research is

needed, however, to determine the exact reason for this relationship and in particular, the meaning of the parameter m .

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5. Appendix I - Measures of Real Networks

Table 1. Real Network Data Used in Paper

Network	n	M	z	ℓ	Category	Type	Source
Protein Interaction	2115	2240	2.1	6.8	biological	Undirected	[23]
Metabolic Network	765	3686	9.6	2.56	biological	Undirected	[2]
Neural Network	307	2359	7.7	3.97	biological	Directed	[24]
Marine Food Web	135	598	4.4	2.05	biological	Directed	[25]
Freshwater Food Web	92	997	10.8	1.9	biological	Directed	[26]
World Trade	179	7697	86.0	1.8	economic	Undirected	[27]
WWW Altavista	203549046	2130000000	10.5	16.18	information	Directed	[28]
Wikipedia	434000	8500000	19.6	4.9	information	Directed	[22]
WWW nd.edu	269504	1497135	5.6	11.27	information	Directed	[29]
Sina.com Blogosphere	200339	1803051	9.0	6.84	information	Directed	[9]
Thesaurus	1022	5103	5.0	4.87	information	Directed	[30]
Cyworld (Korean social networking site)	12048146	190589667	31.6	3.2	social	Undirected	[31]
Biology Coauthors	1520521	11803064	15.5	4.92	social	Undirected	[3, 4]
Film Actors	449913	25516482	113.4	3.48	social	Undirected	[32, 15]
Mixi (Japan social networking site)	360802	1904641	10.6	5.53	social	Undirected	[33]
Math Coauthors	253339	496489	3.9	7.57	social	Undirected	[34, 35]
Email Messages	59912	86300	1.4	4.95	social	Directed	[36]
Physics Coauthors	52909	245300	9.3	6.19	social	Undirected	[3, 4]
Nioki.com (Instant Messaging)	50158	240758	9.6	4.1	social	Undirected	[6]
Pussokram (online dating community)	29341	174662	6.0	4.4	social	Directed	[37]
Email Address Books	16881	57029	3.4	5.22	social	Directed	[38]
Company Directors	7673	55392	14.4	4.6	social	Undirected	[39, 40]
Marvel Comic Characters	6486	168267	51.9	2.63	social	Undirected	[41]
Brazil Pop	5834	507005	173.8	2.3	social	Undirected	[42]
Rapper Collaboration	5533	57972	21.0	3.9	social	Undirected	[11]
Roman/Greek Myth Characters	1637	8938	5.5	3.47	social	Directed	[43]
Jazz Collaboration	1275	38326	60.1	2.79	social	Undirected	[8]
News Article Topics	459	2763	12.0	2.98	social	Undirected	[44]
Dolphin Network	64	159	5.0	3.36	social	Undirected	[45]
Electronic Circuits	24097	53248	4.4	11.05	technological	Undirected	[46]
Internet AS	10697	31992	6.0	3.31	technological	Undirected	[47]
Power Grid	4941	7594	3.1	18.99	technological	Undirected	[15]
Warsaw Public Transport	1530	4406	5.8	19.62	technological	Undirected	[48]
Peer to Peer Networks (Gnutella)	880	1296	2.9	4.28	technological	Undirected	[49, 50]
Indian Railways	587	19603	66.8	2.16	technological	Undirected	[51]
Ammonia Reaction Process	505	759	3.0	7.76	technological	Undirected	[52]
Austria Airports	133	1518	11.4	2.383	technological	Directed	[53]
China Airports	128	2304	36.0	2.07	technological	Undirected	[10]
India Airports	79	442	5.6	2.26	technological	Directed	[54]